ON A TRUTH DEFINITION AND COMPLETENESS OF THE EXTENDED SYSTEM OF PA

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Abstract. There are the recursive total functions e and t such that $\Delta_0$sentence of PA is enumerated by $e =$"x-th $\Delta_0$sentence" and true $\Delta_0$sentence of PA enumerated by $e \circ t$ such that $e(t(x)) =$"x-th true $\Delta_0$sentence". For recursive predicate with one variable $P(x)$, there is the recursive total function f such that $P(x) \leftrightarrow e(t(x))$. So we extend a prenex form to be $\forall x e(f(x))$, then every prenex form of PA is included in them. And if we suppose the set of all recursive total function, then truth definition and decidability of $\forall x e(f(x))$ are reduced to problems of functional equality of recursive functions.

1. Introduction

The extended system denoted EA is obtained from PA by adding the function $y = A(x)$ and defining the enlarged prenex forms. N is the set of all natural numbers and recursive total function is denoted by RT function. RTF is the set of all RT functions. We suppose $A(n)$ is a Gödel number of $\Pi_2$sentence $\forall x \exists ! y P_n(x, y)(\leftrightarrow y = f_n(x))$ for every $n \in N$ and $\forall x \exists ! y P_n(x, y)$ is chosen one from many sentences such that $\forall x \exists ! y P_n(x, y)(\leftrightarrow y = f_n(x))$. Introduced as a sort of choice function $y = A(x)$ itself is able to be defined by some $\Pi_2$sentence of EA. So the function $y = A(x)$ is not only a choice function but also a function of EA after defining prenex forms of EA. (see later Remark 2.7.) Hence the function must be regarded as an oracle, called A. So recursive means A-recursive in EA, but We use simply recursive as A-recursive. The set of all RT function is not r.e but it is r.e in EA. We define the prenex form of EA. $\Delta_0(= \Pi_0 = \Sigma_0)$ sentence of EA is that of PA with all numerals A(n) for all $n \in N$. Every Δ_0 sentence of EA is enumerated by some RT function $e(\Delta_0)(= e(\Pi_0) = e(\Sigma_0))$. For any natural number $n(> 0)$, $\Pi_n(\Sigma_n)$sentences of EA are defined inductively later. We describe the case of $\Pi_1$sentence precisely, $\Pi_1$sentence of EA is defined as follow $R\big(e(\Delta_0) \circ f_m(n)\big) \leftrightarrow P(n)$. (where R is restoration of formula from Gödel number.) For example, F is any formula and $g(F)$ is a Gödel number of F then $R(g(F))$ is F. $\Pi_1$sentence of EA is formulated $\Pi_1(f_m) \equiv \forall x P(x)$. There is the RT function $y = t(x)$ such that an image of $e(\Delta_0) \circ t$ are Gödel numbers of all true $\Delta_0$sentences. Any true $\Pi_1$sentence is represented $\Pi_1(t \circ f_n)$ for some RT function f_n, so given $\Pi_1$sentence $\Pi_1(f_m)$ is treu then there exists some RT function f_n such that $t \circ f_n = f_m$. Problem of rest is when $t \circ f_n = f_m$ hold? This resolution is defining $\Pi_2$sentences which define the function $y = 0$. The above story is the scenario of truth definition and completeness of EA. EA is not axiomatized system.

2. truth definition and completeness of EA

Definition 2.1. S is a set of sentences and there is a recursive injection $G : S \to N$, then We regard $G(a)$ as Gödel number of $a(\in G)$ in EA.

Definition 2.2. RT function which enumerates a member of set $G(S)$ is denoted $e(S)$.

Definition 2.3. The set of all true(false) sentences of S is denoted $TS(FS)$.

Definition 2.4. A function which enumerates a member of the set $Nt = \{n \in N | \ e(S)(n) \in G(TS)\}$ is denoted t. A function which enumerates a member of the set $Nf = \{n \in N| \ e(S)(n) \in G(FS)\}$ is denoted f.

Remark 2.5. $e(S) \circ t$ enumerates a member of $G(TS)$ and $e(S) \circ f$ enumerates a member of $G(FS)$. Gödel number in EA and it in PA is different from each other. Both is regarded as in definition2.1 and denoted $G(s)$ for a sentence s. R is restoration from Gödel number to symbols. $R\big(G(s)\big) = s$.

Theorem 2.6. *All of true prenex form and all of false prenex form is defined in EA and a given prenex form is decidable in EA.*

Proof. Result is almost depend on the definition of EA. The set of all $\Pi_n(\Sigma_n)$sentence of EA is denoted by $\Pi_n(\Sigma_n)$respectively. $\Pi_n(\Sigma_n)$sentences of EA are enumerated by some RT function $e(\Pi_n)(e(\Sigma_n))$ (where an image of $e(\Pi_n)(e(\Sigma_n))$ is all of Gödel number of $\Pi_n(\Sigma_n)$sentences of EA) then every $\Pi_{n+1}(\Sigma_{n+1})$ sentence of EA is obtained from $\Pi_n(\Sigma_n)$sentence of EA by designating some ordered subset which is the image of the function $e(\Sigma_n) \circ f_m, e(\Pi_n) \circ f_m$, (where f_m is some RT function) respectively. At first every $\Delta_0$sentence of EA is the extension of PA by adding all of $A(n)$ for every $n \in N$ as numerals. So omitting these numerals, every $\Delta_0$sentence of EA is same of PA and the result may be obtained from this omitting case. $\Pi_n(\Sigma_n)$ is r.e in EA by existence the function $y = A(x)$. We may assume that there exists the set of all Gödel numbers of $\Pi_n(\Sigma_n)$. We define the set $\Pi_n(\Sigma_n)$ inductively as follows:

$$\Pi_{n+1} = \big\{\Pi_{n+1}(f_m)|\ f_m \in \mathrm{RTF}\big\}, \quad \text{where} \quad \Pi_{n+1} \equiv \forall x R\big(e(\Sigma_n) \circ f_m)(x)\big),$$
$$\Sigma_{n+1} = \big\{\Sigma_{n+1}(f_m)|\ f_m \in \mathrm{RTF}\big\}, \quad \text{where} \quad \Sigma_{n+1} \equiv \exists x R\big(e(\Pi_n) \circ f_m)(x)\big).$$

For $n \in N$, $\Pi_n(\Sigma_n)$sentences of PA are included in $\Pi_n(\Sigma_n)$sentences of PA respectively. Put $B_k = \bigcup_{n<\infty} B(n,k)$ (where $B_k = \{f_m \in \mathrm{RTF}|\ (e(\Pi_k) \circ f_m)(n)\}$), then We may define inductive as follows:

$$T\Sigma_{k+1} = \big\{\Sigma(e(\Pi_k) \circ f_m)|\ f_m \in B_k\big\},$$
$$T\Pi_{k+1} = \big\{\Pi(e(\Pi_k) \circ t \circ f_m)|\ f_m \in \mathrm{RTF}\big\}.$$

$F\Sigma_n(F\Pi_n)$ is obtained from a negation of $T\Sigma_n(T\Pi_n)$ respectively. Especially $T\Sigma_1(0) = \{\Sigma(e(\Pi_0 \circ f_m))|\ f_m \in B(0,0)\} = \{\Sigma(e(\Sigma_1) \circ t_0 \circ f_m)|\ f_m \in \mathrm{RTF}\}$ for some RT function t_0, e.g, $e(\Sigma_1 \circ t_0)$ enumerates an element of $T\Sigma_1(0)$. $T\Pi_2(0) = \{\Pi(e(\Sigma_1 \circ t_0 \circ f_m))|\ f_m \in \mathrm{RTF}\}(\subset T\Pi_2 = \{\Pi(e(\Sigma_1 \circ t \circ f_m))|\ f_m \in \mathrm{RTF}\})$ is the set of all $\Pi_2$sentences of EA which define the constant function $y = 0$. When given two RT functions $y = f_i(x)$ and $y = f_j(x)$, $y = f_i(x) \dot{-} f_j(x)$ is a RT function and this function is defined by some $\Pi_2$sentence θ. $f_i(x) = f_j(x)$ hold for every $x \in N$ if and only if $\Pi_2$sentence θ is a member of $T\Pi_2(0)$. Given $\Pi_n(f_m)$ is true if and only if there exists some RT function $f_n(x)$ such that $f_m(x) = t \circ f_n(x)$ hold for every $x \in N$ and given $\Sigma_n(f_m)$ is true if and only if $f_m(x)$ is a member of B_{n-1}. In EA, all of the set $T\Pi_n, F\Pi_n, T\Sigma_n, F\Sigma_n$ are r.e sets, so above conditions are decidable in EA. □

Remark 2.7. The set $T\Pi_2$ contains a trivial sentence which defines the function $y = A(x)$ in the above abstract. There is the series of true $\Sigma_1$sentences $S(n) \leftrightarrow \exists x C_n(x)$ such that

$$C_n \leftrightarrow R\big(e(\Pi_0 \circ f_k)(x)\big) \leftrightarrow x = A(n)$$

for some $f_k \in \mathrm{RTF}$ and $(e(\Sigma_1) \circ t \circ f_m)(n) = G(S(n))$ for every $n \in N$ for some $f_m \in \mathrm{RTF}$, then we can define $e(\Sigma_1) \circ t \circ f_m) \in T\Pi_2$ which defines the function $y = A(x)$. The set consisting of every true $\Pi_2$sentences $\forall x \exists! y P(x,y)(\leftrightarrow y = f_n(x))$ for every $n \in N$ of PA (in the above abstract) is a proper subset of the set $T\Pi_2 = \{\Pi(e(\Sigma_1) \circ t \circ f_m)|\ f_m \in \mathrm{RTF}\}$, then true $\Pi_2$sentences $\forall x \exists! y P(x,y)$ of EA defines more functions than of RTF. But we must think that every RT function of RTF is defined by true $\Pi_2$sentences of PA. And every $\Pi_2$sentence of PA is regarded as $\Pi_2$sentence of EA by interpretation. What we think about an extension of EA by replacing RTF with the set of all functions defined by true $\Pi_2$sentences of EA is next step. For example, we start from a more small set of functions and to think iterated extensions of EA is interesting them. But we do not deal with it here.

3. Discussion and Summary

When a computer compute somewhat, the computer itself moves by physical rules, so running result may be decidable by physical rules. For example, solutions of some differential equation in physics may decide the computed result by computer. The author expects a computer made of mathematical devices such that gives a solution of the halting problem by mathematics of devices itself.

From the above discussion if there is the set C of functions on N such that closed under the composition operating (e.g. C is a groupoid by composition op functions) and the following condition holds, then we obtain the same result in here with RTF replaced by C.

Condition

(1) For given $f, t \in C$, it is able to judge $f = t \circ g$ for some $g \in C$.
(2) Every prenex forms of PA is regarded as the prenex form of C by interpreting.

If C does not hold (2), then similarly result is established in the system defined by C. An elements of C is not necessary recursive but we may be restricted in recursive total functions naturally from above discussions.

Corollary 3.1. *In fact trivially, it is sufficient that we may take as one of C, generators set of commutative ring RTF (groupoid by composite) and quotient set of RTF by unification of a pair such that computable from one to another. ($f \sim g \leftrightarrow f$ is g recursive and g is f recursive.)*

Because an element of C is defined logically, so it is hard to be independent from recursive or computability. Analysis of $\Pi_2$sentences of EA which define functions may be giving some progression of functions system of EA itself. But the author think it has to be more strongly assumptions to carrying out its analysis. So C is interested in the cases that induce extensions of sentences to be other directions under strongly set theoretic assumption. For example, function rings which include RTF are dealt by working mathematicians.

References

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